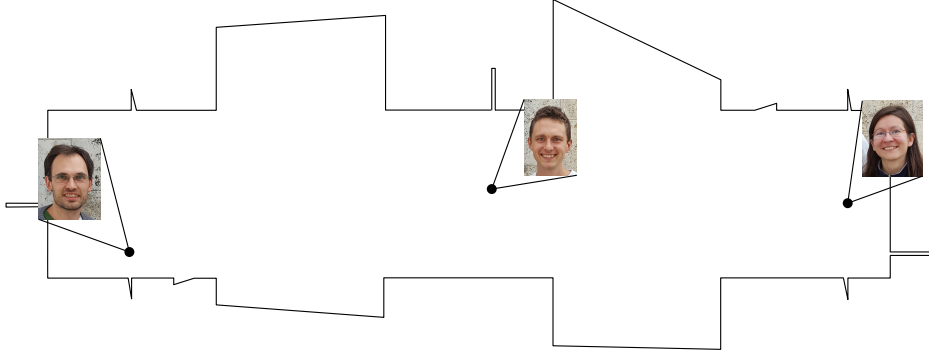


Irrational Guards are Sometimes Needed

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Till, Mikkel, Anna meticulously guarding the polygon: a little irrational, but pretty optimal.

Abstract

In this paper we study the *art gallery problem*, which is one of the fundamental problems in computational geometry. The objective is to place a minimum number of guards inside a simple polygon such that the guards together can see the whole polygon. We say that a guard at position x sees a point y if the line segment xy is fully contained in the polygon.

Despite an extensive study of the art gallery problem, it remained an open question whether there are polygons given by integer coordinates that require guard positions with irrational coordinates in any optimal solution. We give a positive answer to this question by constructing a *monotone* polygon with integer coordinates that can be guarded by three guards only when we allow to place the guards at points with irrational coordinates. Otherwise, four guards are needed. By extending this example, we show that for every n , there is polygon which can be guarded by $3n$ guards with irrational coordinates but need $4n$ guards if the coordinates have to be rational. Subsequently, we show that there are rectilinear polygons given by integer coordinates that require guards with irrational coordinates in any optimal solution.

1 Introduction

For a polygon \mathcal{P} and points $x, y \in \mathcal{P}$, we say that x *sees* y if the interval xy is contained in \mathcal{P} . A *guard set* S is a set of points in \mathcal{P} such that every point in \mathcal{P} is seen by some point in S . The points in S are called *guards*. The *art gallery problem* is to find a minimum cardinality guard set for a simple polygon \mathcal{P} on n vertices. The polygon \mathcal{P} is considered to be filled, i.e., it consists of a closed polygonal curve in the plane and the bounded region enclosed by this curve.

This classical version of the art gallery problem has been originally formulated in 1973 by Victor Klee (see the book of O'Rourke [24, page 2]). It is often referred to as the *interior-guard art gallery problem* or the *point-guard art gallery problem*, to distinguish it from other versions that have been introduced over the years.

In 1978, Steve Fisk provided an elegant proof that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary to guard a polygon with n vertices [18]. Five years earlier, Victor Klee had posed this question to Václav Chvátal, who soon gave a more complicated solution [12]. Since then, the art gallery problem has been extensively studied, both from the combinatorial and the algorithmic perspective. Most of this research, however, is not focused directly on the classical art gallery problem, but on its numerous versions, including different definitions of visibility, restricted classes of polygons, different shapes of the guards, restrictions on the positions of the guards, etc. For more detailed information we refer the reader to the following surveys [20, 24, 29, 31].

Despite extensive research on the art gallery problem, no combinatorial algorithm for finding an optimal solution, or even for deciding whether a guard set of a given size k exists, is known. The only exact algorithm is attributed to Micha Sharir (see [15]), who has shown that in $n^{O(k)}$ time one can find a guard set consisting of k guards, if such a guard set exists. This result is obtained by using standard tools from real algebraic geometry [3], and it is not known how to find an optimal solution without using this powerful machinery (see [4] for an analysis of the very restricted case of $k = 2$). To stress this even more: Without the tools from algebraic geometry, we would not know if it is decidable whether a guard set of size k exists or not! Some recent lower bounds [6] based on the exponential time hypothesis suggest that there might be no better exact algorithms than the one by Sharir.

To explain the difficulty in constructing exact algorithms, we want to emphasize that it is *not* known whether the decision version of the art gallery problem (i.e., the problem of deciding whether there is a guard set consisting of k guards, where k is a parameter) lies in the complexity class NP, even with the algorithm by Sharir. While NP-hardness and APX-hardness of the art gallery problem have been shown for different classes of polygons [7, 16, 21, 22, 25, 28, 30], the question of whether the point-guard art gallery problem is in NP remains open. A simple way to show NP-membership would be to prove that there always exists an optimal set of guards with *rational* coordinates of polynomially bounded description.

Indeed, Sándor Fekete posed at MIT in 2010 and at Dagstuhl in 2011 an open problem, asking whether there are polygons requiring irrational coordinates in an optimal guard set [1, 17]. The question has been raised again by Günter Rote at EuroCG 2011 [26]. It has also been mentioned by Rezende *et al.* [13]: “it remains an open question whether there are polygons given by rational coordinates that require optimal guard positions with irrational coordinates”. A similar question has been raised by Friedrichs *et al.* [19]: “[...] it is a long-standing open problem for the more general Art Gallery Problem (AGP): For the AGP it is not known whether the coordinates of an optimal guard cover can be represented with a polynomial number of bits”.

Our results. We answer the open question of Sándor Fekete, by proving the following main result of our paper. Recall that a polygon \mathcal{P} is called *monotone* if there exists a line l such that every line orthogonal to l intersects \mathcal{P} at most twice.

Theorem 1. *There is a simple monotone polygon \mathcal{P} with integer coordinates of the vertices such that*

- (i) \mathcal{P} can be guarded by 3 guards placed at points with irrational coordinates, and
- (ii) an optimal guard set of \mathcal{P} with guards at points with rational coordinates has size 4.

We then extend this result, by providing a family of polygons for which the ratio between the number of guards in an optimal solution restricted to guards at rational positions, to the number of guards in an optimal solution allowing irrational guards, is $4/3$.

Theorem 2. *There is a family of simple polygons $(\mathcal{P}_n)_{n \in \mathbb{Z}_+}$ with integer coordinates of the vertices such that*

- (i) \mathcal{P}_n can be guarded by $3n$ guards placed at points with irrational coordinates, and
- (ii) an optimal guard set of \mathcal{P}_n with guards at points with rational coordinates has size $4n$.

Moreover, the coordinates of the points defining the polygons \mathcal{P}_n are polynomial in n .

We show that the phenomenon with guards at irrational coordinates occurs also in the important class of rectilinear polygons.

Theorem 3. *There is a rectilinear polygon \mathcal{P}_R with vertices at integer coordinates satisfying the following properties.*

- (i) \mathcal{P}_R can be guarded by 9 guards if we allow placing guards at points with irrational coordinates.
- (ii) An optimal guard set of \mathcal{P}_R with guards at points with rational coordinates has size 10.

Other related work. The art gallery problem has been studied from the perspective of approximation algorithms. Efrat and Har-Peled [15] gave a randomized polynomial time algorithm for finding a guard set S where the guards are restricted to a very fine grid Γ . To be more precise, if coordinates of the vertices of the input polygon \mathcal{P} are given by positive integers and L is the largest such integer, then Γ can be defined as the points in $L^{-20} \cdot \mathbb{Z}^2 \cap \mathcal{P}$. Let $OPT_{\text{grid}} \subset \Gamma$ be a guard set with a minimum number of guards under this restriction. The algorithm of Efrat and Har-Peled yields an $O(\log |OPT_{\text{grid}}|)$ -approximation for this problem. However, it remained open whether OPT_{grid} is an approximation of an optimal unrestricted guard set OPT . Bonnet and Miltzow [5] filled this gap by showing that under a general position assumption $|OPT_{\text{grid}}| = O(|OPT|)$, which yields the first polynomial time approximation algorithm for simple polygons under this assumption. It is easy to construct a polygon with integer coordinates that forces a guard on the point $(1/3, 1/3)$, which might not lie on the grid, in case that L is not divisible by 3. This implies that OPT_{grid} is not optimal. But this does not rule out that there is a slightly more clever choice of Γ so that OPT_{grid} is indeed optimal. It follows from our Theorem 2 that there are polygons (requiring arbitrarily many guards in an optimal guard set) such that for any choice of $\Gamma \subset \mathbb{Q}^2$, it holds that $|OPT_{\text{grid}}| \geq 4/3 \cdot |OPT|$. No lower bound of this kind has been known before. More general, our result shows that no algorithm which considers only rational points as possible guard positions can achieve an approximation ratio better than $4/3$.

A new line of research focuses on implementing algorithms that are capable of solving instances of the art gallery problem with thousands of vertices, giving a solution which is close to the optimal one, see the recent survey by Rezende *et al.* [13]. They explain that many practical algorithms rely on “routines to find candidates for discrete guard and witness locations.” We show that this technique inevitably leads to sub-optimal solutions unless irrational candidate locations are also considered. We believe that our example and techniques are a good starting point to construct benchmark instances for implementations of art gallery algorithms. Benchmark instances serve to validate the quality of algorithms. Using the same instances when comparing different algorithms makes the results comparable.

A problem related to the art-gallery problem is the *terrain guarding problem*. In this problem, an x -monotone polygonal curve c (i.e., the terrain) is given. The region R above the curve c has to be guarded, and the guards are restricted to lie on c . Similarly as in our problem, a guard x sees a point y if xy is contained in the region R . Although the solution space of the terrain guarding instance is the continuous polygonal curve c , a discretization of the solution space has been recently described by Friedrichs *et al.* [19]. Given a terrain with n vertices at integer position, they describe a set $S \subset \mathbb{Q}$ of size $O(n^3)$, computable in polynomial time, such that there is an optimal guard placement restricted to S . It follows that for the terrain guarding problem the phenomenon with irrational numbers does not appear, and also the decision version of the terrain guarding problem is in NP.

Irrational numbers turn up surprisingly in other areas of computational geometry. One such example is the *nested polytopes problem*. Here, we are given two nested polytopes $S \subseteq P$ and want to find a polytope T with a minimum number of corners such that T is nested between S and P , i.e., $S \subseteq T \subseteq P$. Christikov *et al.* [11] recently gave an example of two nested polytopes $S \subseteq P$ in \mathbb{R}^3 , with all corners at rational coordinates, such that there is a unique polytope T with 5 corners nested between S and P , and T has corners with irrational coordinates. The nested polytopes problem is closely related to *nonnegative*

matrix multiplication, where similar phenomena have been discovered, that a problem defined entirely by rational numbers has an optimal solution requiring irrational numbers [10, 11].

The Structure of the Paper. Section 2 contains the description of a monotone polygon \mathcal{P} with vertices at points with rational coordinates that can be guarded by three guards only if the guards are placed at points with irrational coordinates. In Section 3, we describe the intuition behind our construction, and explain how we have found the polygon \mathcal{P} . The formal proof of Theorems 1 and 2 is then provided in Section 4. In Section 5, we present the rectilinear polygon \mathcal{P}_R from Theorem 3 requiring guards with irrational coordinates in an optimal guard set. Finally, in Section 6 we suggest some open problems for future research.

2 The Polygon

In Figure 1 we present the polygon \mathcal{P} . In Section 4 we will prove that \mathcal{P} can be guarded by three guards only when we allow the guards to be placed at points with irrational coordinates.

The polygon \mathcal{P} is constructed as follows. We start with a *basic rectangle* $[0, 20] \times [0, 4] \subset \mathbb{R}^2$. Then, we append to it six *triangular pockets* (colored with green in the figure), which are triangles defined by the following coordinates

$$\begin{aligned} T_t^\ell &: \{(2, 4), (2, 4.5), (2.1, 4)\}, & T_b^\ell &: \{(2, 0), (2, -0.5), (1.9, 0)\}, \\ T_t^m &: \{(16\frac{5}{6}, 4), (17\frac{2}{6}, 4.15), (17\frac{2}{6}, 4)\}, & T_b^m &: \{(3.5, 0), (3, -0.15), (3, 0)\}, \\ T_t^r &: \{(19, 4), (19, 4.5), (19.1, 4)\}, & \text{and } T_b^r &: \{(19, 0), (19, -0.5), (18.9, 0)\}. \end{aligned}$$

Next, we append three *rectangular pockets* (colored with blue in the figure, for practical reasons these pockets are drawn in the figure shorter than they actually are), which are rectangles defined in the following way.

$$R_\ell: [-10, 0] \times [1.7, 1.8], \quad R_r: [20, 30] \times [0.5, 0.6], \quad \text{and } R_m: [10.5, 10.6] \times [4, 8].$$

Last, we append four *quadrilateral pockets* (colored with red in the figure), which are defined by points with the following coordinates

$$\begin{aligned} \text{Top-left pocket } P_t^\ell & \quad \{(4, 4), \quad (4, \frac{280}{47}), \quad (8, \frac{294}{47}), \quad (8, 4)\} \\ \text{Top-right pocket } P_t^r & \quad \{(12, 4), \quad (12, \frac{2486}{375}), \quad (16, \frac{1776}{375}), \quad (16, 4)\} \\ \text{Bottom-left pocket } P_b^\ell & \quad \{(4, 0), \quad (4, -\frac{12}{19}), \quad (8, -\frac{18}{19}), \quad (8, 0)\} \\ \text{Bottom-right pocket } P_b^r & \quad \{(12, 0), \quad (12, -\frac{34}{21}), \quad (16, -\frac{36}{21}), \quad (16, 0)\}. \end{aligned}$$

The polygon \mathcal{P} is clearly monotone. We will denote by e_t^ℓ , e_t^r , e_b^ℓ , and e_b^r the non-axis-parallel edge within each of the four quadrilateral pockets, respectively.

3 Intuition

In this section, we explain the key ideas behind the construction of the polygon \mathcal{P} . Our presentation is informal, but it resembles the work process that lead to the construction of \mathcal{P} more than the formal proof of Theorem 1 in Section 4 does. Here we omit all “scary” computations and focus on conveying the big picture. In the end of this section, we also explain how we actually constructed the polygon \mathcal{P} .

Define a *rational point* to be a point with two rational coordinates. An *irrational point* is a point that is not rational. A *rational line* is a line that contains two rational points. An *irrational line* is a line that is not rational.

Forcing a Guard on a Line Segment. Consider the drawing of the polygon \mathcal{P} in Figure 1. We will now explain an idea of how three pairs of triangular pockets, (T_t^ℓ, T_b^ℓ) , (T_t^m, T_b^m) , and (T_t^r, T_b^r) , can enforce three guards on three line segments within \mathcal{P} .

Consider the two triangular pockets in Figure 2a. The blue line segment contains one edge of each of these pockets, and the interiors of the pockets are at different sides of the line segment. A guard which sees the point t must be placed within the orange triangular region, and a guard which sees b must be placed within the yellow triangular region. Thus, a single guard can see both t and b only if it is on the blue line segment tb , which is the intersection of the two regions.

Consider now the case that we have k pairs of triangular pockets, and no two regions corresponding to different pairs of pockets intersect. In order to guard the polygon with k guards, there must be one guard on the line segment corresponding to each pair. Our polygon \mathcal{P} has three such pairs of pockets (see Figure 2b), and it can be checked that the corresponding regions do not intersect.

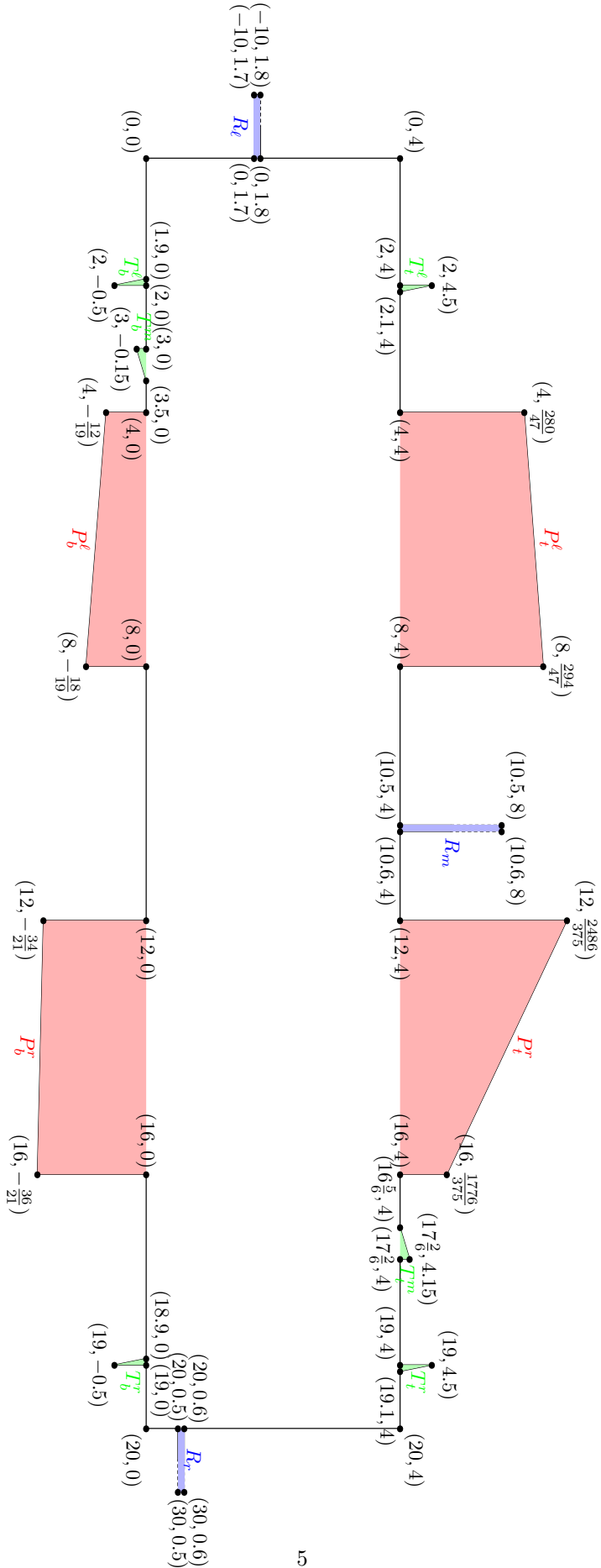
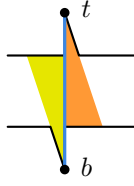
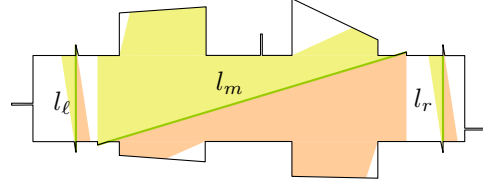


Figure 1: The polygon \mathcal{P} . We will show that \mathcal{P} can be guarded by three guards only when we allow the guards to be placed at points with irrational coordinates. For practical reasons, the blue rectangular pockets are drawn shorter than they actually are.



(a) The only way that one guard can see both t and b is when the guard is on the blue line.



(b) The only way to guard the polygon with three guards requires one guard on each of the green lines l_ℓ, l_m, l_r .

Figure 2: Forcing guards to lie on specific line segments.

Notice that in this way we can only enforce a guard to be on a rational line, as the line contains vertices of the polygon, which are rational points.

Restricting a Guard to a Region Bounded by a Curve. For the following discussion, see the Figure 3 and notation therein. We want to guard the polygon from Figure 3 using two guards, g_1 and g_2 . We assume that g_1 is forced to be on the blue vertical line l .

Consider some position of g_1 on l , such that g_1 can see at least one point of the top edge e_t of the top quadrilateral pocket, and at least one point of the bottom edge e_b of the bottom quadrilateral pocket. Let p_t and p_b denote the leftmost points seen by g_1 on e_t and e_b , respectively. Observe that p_t moves to the right if g_1 moves up, and to the left if g_1 moves down. The point p_b behaves in the opposite way when g_1 is moved. Consider some fixed position of g_1 on the blue line, and the corresponding positions of p_t and p_b . Let b be the bottom right corner of the top pocket, and d the top right corner of the bottom pocket. Let i be the intersection point of the line containing p_t and b , with the line containing p_b and d . The points b, d, i define a triangular region Δ . It is clear that if we place the guard g_2 anywhere inside Δ , then g_1 and g_2 will together see the entire polygon. On the other hand, if we place g_2 to the right of Δ , then g_1 and g_2 will not see the entire polygon, as some part of the top or the bottom pocket will not be seen.

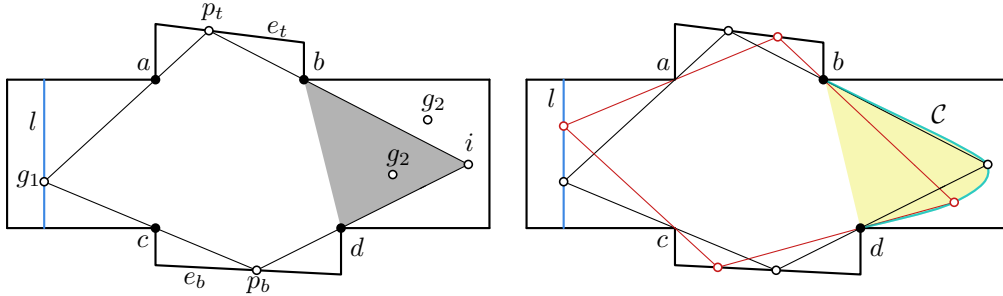


Figure 3: Left: The guard g_2 must be inside the triangular region (or to the left of it) in order to guard the entire part of the polygon that is not seen by g_1 . Right: All possible positions of the point i define a simple curve \mathcal{C} .

Now, let us move the guard g_1 along the blue line. Each position of g_1 yields some intersection point i . We denote the union of all these intersection points by \mathcal{C} (see the right picture in Figure 3). It is easy to see that \mathcal{C} is a simple curve. We can compute a parameterization of \mathcal{C} since we have described how to construct the point i as a function of the position of g_1 .

Note that g_2 sees a larger part of *both* pockets if it is moved horizontally to the left and a smaller part of *both* pockets if it is moved horizontally to the right. Consider a fixed position of g_2 on or to the right of the segment bd . Let g'_2 be the horizontal projection of g_2 on \mathcal{C} . Let g_1 be the unique position on the blue line such that g_1 and g'_2 see all of the polygon. If g_2 is to the left of \mathcal{C} , g'_2 sees less of the pockets than g_2 , so g_1 and g_2 can together see everything. If g_2 is to the right of \mathcal{C} , g_2 sees less of the pockets than g'_2 and neither the top nor the bottom pocket are completely guarded by g_1 and g_2 . For any higher placement of g_1 even less of the top pocket is guarded and for any lower placement of g_1 even less of the bottom pocket is guarded. Thus, there exists no placement of g_1 such that both pockets are completely guarded by g_1 and g_2 . We summarize our reasoning in the following observation.

Observation 1. Consider a fixed position of g_2 on or to the right of the segment bd . There exists a position of g_1 on l such that the entire polygon is seen by g_1 and g_2 if and only if g_2 lies on or to the left of the curve \mathcal{C} .

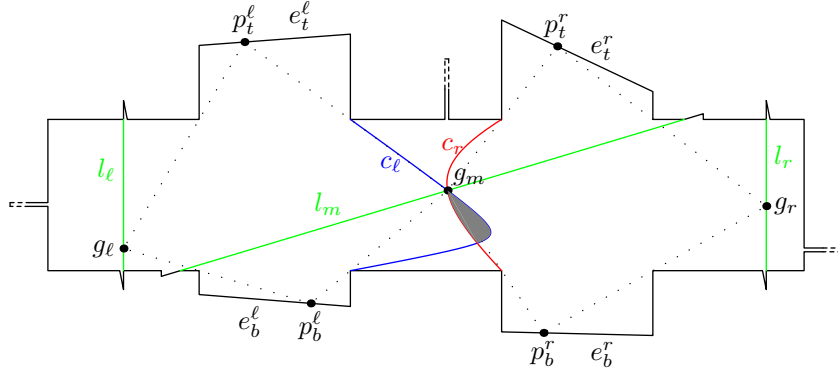


Figure 4: The polygon \mathcal{P} .

Restricting a Guard to a Single (Irrational) Point. For this paragraph, let us consider the polygon \mathcal{P} introduced in Section 2, and consider a guard set for \mathcal{P} consisting of three guards. The polygon \mathcal{P} is drawn again in Figure 4, together with additional labels and information. The three guards g_ℓ, g_m, g_r are forced by the triangular pockets to lie on the three green lines l_ℓ, l_m, l_r , respectively. Additionally, the three rectangular pockets R_ℓ, R_m, R_r force the guards to lie within one of three short intervals within each line. (These properties of our construction will be discussed in more detail in Section 4.) With these restrictions, we will show that for the three guards to see the whole polygon, it must hold that the guards g_ℓ and g_m can together see the left pockets P_t^ℓ and P_b^ℓ , and the guards g_m and g_r can together see the right pockets P_t^r and P_b^r .

Then, the curve c_ℓ bounds from the right the feasible region for the guard g_m , such that g_ℓ and g_m can together see the left pockets P_t^ℓ and P_b^ℓ . Similarly, the curve c_r bounds from the left the feasible region for the guard g_m , such that g_r and g_m can together see the right pockets P_t^r and P_b^r . Thus, the only way that g_ℓ, g_m , and g_r can see the whole polygon is when g_m is within the grey region, between c_ℓ and c_r . Our idea is to define the line l_m so that it contains an intersection point of c_ℓ and c_r , and it does not enter the interior of the grey region. A simple computation with sage [14] outputs equations defining the two curves:

$$c_\ell : 138x^2 - 568xy - 1071y^2 - 3018x + 8828y + 15312 = 0 ,$$

$$c_r : 138x^2 - 156xy - 356y^2 - 1791x + 3296y + 1620 = 0 .$$

See Appendix A for the sage code for this computation. It can be checked, even by hand, that the point

$$p = (3.5 + 5\sqrt{2}, 1.5\sqrt{2}) \approx (10.57, 2.12)$$

lies on both curves, and also on the line $l_m = \{ (x, y) : y = 0.3x - 1.05 \}$. Therefore, p is a feasible (and at the same time irrational) position for the guard g_m . Moreover, by plotting c_ℓ , c_r , and l_m in \mathcal{P} as in Figure 4, we get an indication that as we traverse l_m from left to right, at the point p we exit the area where g_m and g_l can guard together the two left pockets, and at the same time we enter the area where g_m and g_r can guard together the two right pockets. Thus, the only feasible position for the guard g_m is the irrational point p . A formal proof will be given in Section 4.

Searching for the Polygon. The simplicity of the ideas behind our construction does not reflect the difficulty of finding the exact coordinates for the polygon \mathcal{P} . The reader might for instance presume that most other choices of horizontal pockets would work, if the line l_m is changed accordingly. However, this is not the case.

It is easy to construct the pockets so that the corresponding curves c_ℓ and c_r intersect at some point p . We expect p to be an irrational point in general, since the curves c_ℓ and c_r are defined by two second

degree polynomials, as indicated above. In our construction, we need to force g_m to be on a line l_m containing p , but we can only force g_m to be on a rational line. Hence, we require the existence of a rational line l_m that contains p .

As any two rational lines intersect in a rational point, there can be at most one rational line containing the irrational point p . Moreover, there exists a rational line containing p if and only if $p = (r_1 + r_2\alpha, r_3 + r_4\alpha)$ for some $r_1, r_2, r_3, r_4 \in \mathbb{Q}$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational number. The equation of the rational line containing p is then $y = \frac{r_4}{r_2} \cdot x + (r_3 - r_1 \cdot \frac{r_4}{r_2})$. We say that this line *supports* p . Therefore, we should not hope that the intersection point of the curves c_ℓ and c_r defined by arbitrarily chosen pockets will have a supporting line. Our main idea to overcome this problem has been to reverse-engineer the polygon, after having chosen the positions of the guards. We chose three irrational guards, all with supporting rational lines, and then defined the pockets so that g_m automatically became the intersection point between the curves c_ℓ and c_r associated with the pockets.

We chose all three guards to have coordinates of the form $(r_1 + r_2\sqrt{2}, r_3 + r_4\sqrt{2})$ for $r_1, r_2, r_3, r_4 \in \mathbb{Q}$. Assume, for the ease of presentation, that we already know that we can end up with a polygon described as follows. (In our initial attempts, our polygons were much less regular.) The polygon should consist of the rectangle $R = [0, 20] \times [0, 4]$ with some pockets added. We would like the pockets to extrude vertically from the horizontal edges of R such that the pockets meet R along the segments $(4, 0)(8, 0)$, $(12, 0)(16, 0)$, $(4, 4)(8, 4)$, and $(12, 4)(16, 4)$, respectively.

We now explain the technique for constructing the bottom pocket to the left which should extrude from R vertically downwards from the corners $(4, 0)$ and $(8, 0)$. We have to define the edge e_b^ℓ , which is the bottom edge in the pocket. We want p_b^ℓ to be a point on e_b^ℓ such that g_ℓ can only see the part of e_b^ℓ from p_b^ℓ and to the right, whereas g_m can only see the part of e_b^ℓ from p_b^ℓ and to the left. Therefore, we define p_b^ℓ to be the intersection point between the line containing g_ℓ and $(4, 0)$, and the line containing g_m and $(8, 0)$. It follows that p_b^ℓ is of the form $(r_1 + r_2\sqrt{2}, r_3 + r_4\sqrt{2})$ for some $r_1, r_2, r_3, r_4 \in \mathbb{Q}$. Hence, there is a unique rational line l supporting p_b^ℓ , and e_b^ℓ must be a segment on l . We therefore need that both of the points $(4, 0)$ and $(8, 0)$ are above l , since otherwise we do not get a meaningful polygon. However, this is not the case for arbitrary choices of the guards g_ℓ and g_m . The other pockets add similar restrictions to the positions of the guards.

In the construction we had to take care of other issues as well. In particular, the line l_m which supports the guard g_m cannot enter the grey region between the two curves c_ℓ and c_r , as otherwise the position of g_m would not be unique, and the guard could be moved to a rational point. Also, the three lines l_ℓ, l_m, l_r supporting the three guards g_ℓ, g_m, g_r cannot intersect within the polygon.

We experimented with the construction in GeoGebra [2], where we had the possibility to draw the lines supporting $p_t^\ell, p_b^\ell, p_t^r, p_b^r$ and see how they were changing in an intricate way as we changed the coordinates of the guards. For most choices of the guards and other parts of the polygon, we did not get meaningful results. The great advantage of GeoGebra is that we could continuously vary all parts of the polygon and play around with all parameters, thus gaining an intuitive understanding of various dependencies. After experimenting for a while, we were able to produce feasible examples and then find more appealing examples with simpler coordinates etc. In particular, it was important to us that many edges of the polygon are axis-parallel, so that we could easier derive from our example a rectilinear polygon with the same property, i.e., that the optimal guard set requires points with irrational coordinates.

4 Proof of Theorems 1 and 2

Basic observations. Recall the construction of the polygon \mathcal{P} as defined in Section 2, and consider a guard set of \mathcal{P} of cardinality at most 3. Let l_ℓ, l_m, l_r be, respectively, the restrictions of the following lines to \mathcal{P} :

$$x = 2, \quad y = 0.3x - 1.05, \quad \text{and} \quad x = 19.$$

As argued in Section 3, the triangular pockets enforce a guard onto each of these lines.

Lemma 4. *Consider any guard set S for \mathcal{P} consisting of at most 3 guards. Then (i) $|S| = 3$, and (ii) there is one guard on each of the lines l_ℓ, l_m, l_r .*

Proof. Each triangular pocket $T_t^\ell, T_b^\ell, T_t^m, T_b^m, T_t^r, T_b^r$ has one vertex which is not on the basic rectangle $[0, 20] \times [0, 4]$. For each triangular pocket, we consider the points in \mathcal{P} that can see that vertex. These positions correspond to the areas pictured in yellow and orange in Figure 2b.

It is straightforward to check that the only positions of guards that can see two such vertices are on the segments l_ℓ, l_m, l_r . Since these segments are non-intersecting, at least three guards are needed to see the whole polygon \mathcal{P} . If there are three guards, then there must be one guard on each of the segments l_ℓ, l_m, l_r . \square

Now, consider the intervals $i_1 = [0.5, 0.6]$ and $i_2 = [1.7, 1.8]$. Similarly as for the case of triangular pockets, we can show that rectangular pockets R_ℓ, R_m, R_r enforce a guard with an x-coordinate in $[10.5, 10.6]$, and two remaining guards with y-coordinates in i_1 and i_2 .

Lemma 5. *Consider any guard set S for \mathcal{P} consisting of 3 guards. Then one of the guards has an x-coordinate in $[10.5, 10.6]$. For the remaining two guards, one has a y-coordinate in i_1 and the other one in i_2 .*

Proof. From Lemma 4, there must be one guard g_ℓ on l_ℓ , one guard g_m on l_m , and the last guard g_r on l_r . Recall that the rectangular pockets are as follows $R_\ell: [-10, 0] \times [1.7, 1.8]$, $R_r: [20, 30] \times [0.5, 0.6]$, and $R_m: [10.5, 10.6] \times [4, 8]$. It is straightforward to check that none of the guards g_ℓ, g_r can see the two top vertices of the pocket R_m . Therefore, the middle guard g_m has to see both these vertices and it must have an x-coordinate in $[10.5, 10.6]$.

Then, as $g_m \in l_m$, the y-coordinate of g_m is in $[2.1, 2.13]$. Therefore, g_m cannot see any of the left vertices of R_ℓ , or any of the right vertices of R_r . These four vertices must be seen by the guards g_ℓ and g_r .

As some guard must see the bottom-left corner of the pocket R_ℓ , it must be placed at a height of at least 1.7. Then, this guard cannot see any of the right vertices of R_r . Therefore, the last guard must see both right vertices of R_r , and its height must be within $i_1 = [0.5, 0.6]$. Then, this guard cannot see any left vertex of the pocket R_ℓ , and the second guard must see both left vertices of the pocket, and its height must be within $i_2 = [1.7, 1.8]$. \square

Dependencies between guard positions. Let $\{g_\ell, g_m, g_r\}$ be a guard set of \mathcal{P} , with $g_\ell \in l_\ell, g_m \in l_m$, and $g_r \in l_r$. We will now analyze dependencies between the positions of the guards that are caused by the horizontal pockets of \mathcal{P} . Recall that the non-axis-parallel edges of these pockets are denoted by e_t^ℓ, e_b^ℓ , and e_b^r .

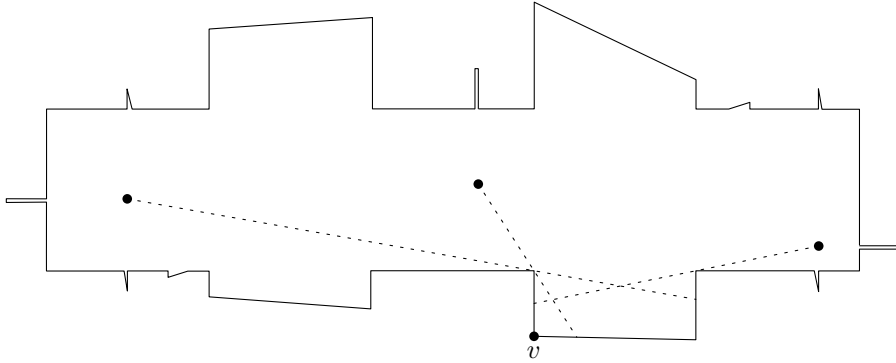
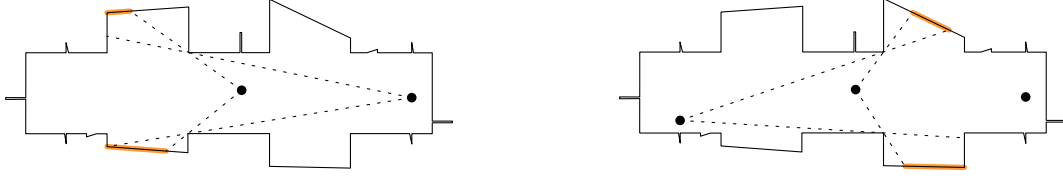


Figure 5: If the guard g_r guards the wrong pocket, no guard can see the vertex v .

Lemma 6. *The y-coordinate of guard g_ℓ is in the interval $i_1 = [0.5, 0.6]$ and the y-coordinate of guard g_r is in the interval $i_2 = [1.7, 1.8]$*

Proof. It is clear that one of the y-coordinate of g_r is either in the interval i_1 or i_2 by the arguments given above. In case that the y-coordinate of g_r is in i_1 , it is easily seen that g_r cannot see the bottom left corner v of the bottom right point, see Figure 5. None of the other guards can see v either — a contradiction. \square

Lemma 7. *The guards g_ℓ and g_m together see all of e_t^ℓ and e_b^ℓ , and the guards g_m and g_r together can see all of e_t^r and e_b^r .*



(a) The part of e_t^ℓ, e_b^ℓ that is seen by g_r is also seen by g_m . (b) The part of e_t^r, e_b^r that is seen by g_ℓ is also seen by g_m .

Figure 6: The left guard is not helpful to guard the right pockets and the right guard is not helpful to guard the left pockets.

Proof. By the construction of \mathcal{P} , it holds that if a guard sees a point on one of the edges $e_t^\ell, e_b^\ell, e_t^r$, and e_b^r , then the guard sees an interval of the edge containing an endpoint of the edge. It now follows that if three guards together see one of these edges, then two do as well. Also note that it is impossible for a single guard to see either of the edges entirely. In order to prove the lemma, it thus suffices to prove that

- The part of e_t^ℓ, e_b^ℓ that is seen by g_r is also seen by g_m .
- The part of e_t^r, e_b^r that is seen by g_ℓ is also seen by g_m .

The Lemma follows from the two statements above. The two statements can be easily checked in Figure 6a and 6b. \square

Computing the unique solution. We can now show that there is only one guard set for \mathcal{P} consisting of three guards. Let us start by computing the right-most possible position of g_m such that g_ℓ and g_m can see together both left pockets.

For the next two lemmas, recall the notation from Figure 4.

Lemma 8. *The maximum x -coordinate of g_m such that g_ℓ and g_m can together see e_t^ℓ and e_b^ℓ is $x = 3.5 + 5\sqrt{2}$. The corresponding position of g_ℓ is $(2, 2 - \sqrt{2})$.*

Proof. Consider the guard g_ℓ at position $(2, h)$. From Lemma 6, we know that $h \in i_1 = [0.5, 0.6]$. We can easily compute

$$p_t^\ell = \left(\frac{908 - 188h}{181 - 47h}, \frac{7}{94} \cdot \frac{908 - 188h}{181 - 47h} + \frac{266}{47} \right)$$

If g_m and g_ℓ together see e_t^ℓ , we know from Lemma 7 that g_m has to be on or below the line containing the vertices $(8, 4)$ and p_t^ℓ , i.e., the line with equation

$$y = \frac{92 - 23h}{-135 + 47h}x + \frac{-1276 + 372h}{-135 + 47h}.$$

As g_m is on the line l_m described by $y = 0.3x - 1.05$, its x -coordinate satisfies

$$0.3x - 1.05 \leq \frac{92 - 23h}{-135 + 47h}x + \frac{-1276 + 372h}{-135 + 47h},$$

i.e.,

$$x \leq \frac{28355 - 8427h}{2650 - 742h}.$$

If g_m and g_ℓ together see e_b^ℓ , then g_m has to be on or above the line containing the vertices $(8, 0)$ and

$$p_b^\ell = \left(\frac{76h + 12}{19h - 3}, -\frac{3}{38} \cdot \frac{76h + 12}{19h - 3} - \frac{6}{19} \right),$$

i.e., the line with equation

$$y = \frac{3h}{19h - 9}x - \frac{24h}{19h - 9}.$$

Hence, the x -coordinate of g_ℓ must satisfy

$$0.3x - 1.05 \geq \frac{3h}{19h - 9}x - \frac{24h}{19h - 9},$$

i.e.,

$$x(1 - h) \leq \frac{81h + 189}{54}.$$

Therefore, since $h < 1$, we must have

$$x \leq \frac{81h + 189}{54 - 54h}.$$

We now know that

$$x \leq \min \left\{ \frac{28355 - 8427h}{2650 - 742h}, \frac{81h + 189}{54 - 54h} \right\}.$$

The first of the two values decreases with h , and the second one increases with h . Therefore the maximum is obtained when

$$\frac{28355 - 8427h}{2650 - 742h} = \frac{81h + 189}{54 - 54h},$$

i.e., for $h = 2 - \sqrt{2}$. The value of x is then $3.5 + 5\sqrt{2}$. The corresponding position of the guard g_ℓ is $(2, h) = (2, 2 - \sqrt{2})$. \square

Similarly, we can compute the left-most possible position of g_m such that g_m and g_r can see together both right pockets. The proof is in the appendix.

Lemma 9. *The minimum x -coordinate of g_m such that g_r and g_m can see both e_t^r and e_b^r is $x = 3.5 + 5\sqrt{2}$. The corresponding position of g_r is $(19, 1 + \frac{\sqrt{2}}{2})$.*

Proof. Consider the guard g_r at position $(19, h)$. From Lemma 6, we know that $h \in i_2 = [1.7, 1.8]$. If g_m and g_r together see e_t^r , we know from Lemma 7 that g_m has to be on or below the line containing the vertices $(12, 4)$ and

$$p_t^r = \left(\frac{4000h - 9768}{250h - 645}, -\frac{71}{150} \frac{4000h - 9768}{250h - 645} + \frac{4616}{375} \right),$$

i.e., the line with equation

$$y = \frac{46h - 184}{250h - 507}x + \frac{448h + 180}{250h - 507}.$$

As g_m is at the line $y = 0.3x - 1.05$, its x coordinate satisfies:

$$0.3x - 1.05 \leq \frac{46h - 184}{250h - 507}x + \frac{448h + 180}{250h - 507},$$

i.e.,

$$x \geq \frac{490h - 243}{20h + 22}.$$

If g_m and g_r together see e_b^r , then g_m has to be on or above the line containing the vertices $(12, 0)$ and

$$p_b^r = \left(\frac{224h - 56}{14h + 1}, -\frac{1}{42} \frac{224h - 56}{14h + 1} - \frac{4}{3} \right),$$

i.e., the line with equation

$$y = \frac{6h}{17 - 14h}x - \frac{72h}{17 - 14h}.$$

Hence, the x -coordinate of g_r must satisfy

$$0.3x - 1.05 \geq \frac{6h}{17 - 14h}x - \frac{72h}{17 - 14h},$$

i.e., $x \geq \frac{34h - 7}{4h - 2}$.

We have to minimize the value of

$$\max \left\{ \frac{490h - 243}{20h + 22}, \frac{34h - 7}{4h - 2} \right\}.$$

When the value of h increases, the first of these two values increases, and the second one decreases. The minimum value is therefore obtained when

$$\frac{490h - 243}{20h + 22} = \frac{34h - 7}{4h - 2},$$

i.e., for $h = 1 + \frac{\sqrt{2}}{2}$. The value of x is then $3.5 + 5\sqrt{2}$. \square

We are now ready to prove our main theorems.

Proof of Theorem 1. Let \mathcal{P} be the polygon constructed as in Section 2, and let S be a guard set for \mathcal{P} consisting of at most 3 guards. From Lemma 4 we have $|S| = 3$, and there is one guard at each of the lines l_ℓ, l_m, l_r . Denote these guards by g_ℓ, g_m, g_r , respectively. From Lemma 7 we know that if g_ℓ, g_m , and g_r together see all of \mathcal{P} , then g_ℓ and g_m must see all of e_t^ℓ and e_b^ℓ , and g_m and g_r must see all of e_t^r and e_b^r . It then follows from Lemmas 8 and 9 that g_m must have coordinates $(3.5 + 5\sqrt{2}, 1.5\sqrt{2}) \approx (10.57, 2.12)$, $g_\ell = (2, 2 - \sqrt{2}) \approx (2, 0.59)$, and $g_r = (19, 1 + \frac{\sqrt{2}}{2}) \approx (19, 1.71)$. Thus, indeed, the guards g_ℓ, g_m , and g_r see the entire polygon \mathcal{P} and are the only three guards doing so.

By scaling \mathcal{P} up by the least common multiple of the denominators in the coordinates of the corners of \mathcal{P} , we obtain a polygon with integer coordinates. This does not affect the number of guards required to see all of \mathcal{P} .

In order to guard \mathcal{P} using four guards with rational coordinates, we choose two rational guards $g'_{m,1}$ and $g'_{m,2}$ on l_m a little bit to the left and to the right of g_m , respectively. The guard $g'_{m,1}$ sees a little more of both of the edges e_t^ℓ and e_b^ℓ than does g_m , whereas $g'_{m,2}$ sees a little more of e_t^r and e_b^r . Therefore, we can choose a rational guard g'_ℓ on l_ℓ close to g_ℓ such that g'_ℓ and $g'_{m,1}$ together see e_t^ℓ and e_b^ℓ , and a rational guard g'_r on l_r with analogous properties. Thus, $g'_\ell, g'_{m,1}, g'_{m,2}, g'_r$ guard \mathcal{P} . \square

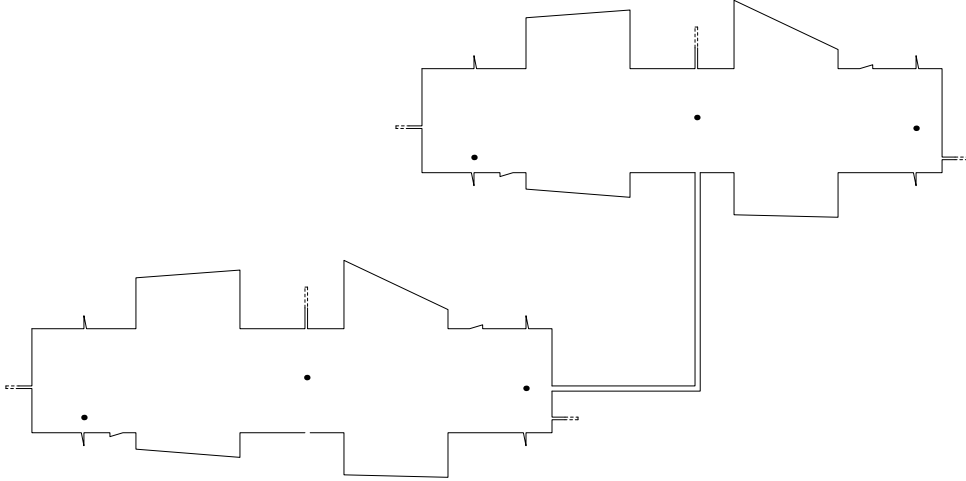


Figure 7: A sketch of a polygon that can be guarded by 6 guards when irrational coordinates are allowed, but needs 8 guards when only rational coordinates are allowed.

Proof of Theorem 2. We will now construct a polygon \mathcal{P}_n that can be guarded by $3n$ guards placed at points with irrational coordinates, but such that when we restrict guard positions to points with rational coordinates, the minimum number of guards becomes $4n$. We start by making n copies of the polygon \mathcal{P} described above, which we denote by $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(n)}$. We connect the copies into one polygon \mathcal{P}_n as follows. Each consecutive pair $\mathcal{P}^{(i)}, \mathcal{P}^{(i+1)}$ is connected by a thin corridor consisting of a horizontal piece $H^{(i)}$ visible by the rightmost guard in $\mathcal{P}^{(i)}$, and a vertical piece $V^{(i)}$ visible to the middle guard in $\mathcal{P}^{(i+1)}$

(see Figure 7 for the case $n = 2$). We can then guard \mathcal{P}_n using $3n$ guards, by placing three guards within each polygon $\mathcal{P}^{(i)}$ in the same way as for \mathcal{P} , i.e., at irrational points.

Now, assume that \mathcal{P}_n can be guarded by at most $4n - 1$ guards. We will show that at least one guard must be irrational. For formal reasons, we define $H^{(0)} = V^{(0)} = H^{(n)} = V^{(n)} = \emptyset$. The horizontal and vertical corridors $H^{(i)}$ and $V^{(i)}$, for $i \in \{0, \dots, n\}$, intersect at a rectangular area $B^{(i)} = H^{(i)} \cap V^{(i)}$ which we call a *bend*. For $i \in \{1, \dots, n-1\}$, the bend $B^{(i)}$ is non-empty and visible from both polygons $\mathcal{P}^{(i)}$ and $\mathcal{P}^{(i+1)}$. Define the *extension* of $\mathcal{P}^{(i)}$, denoted by $E(\mathcal{P}^{(i)})$, to be the union of $\mathcal{P}^{(i)}$ and the adjacent corridors excluding the bends, i.e., $E(\mathcal{P}^{(i)}) = \mathcal{P}^{(i)} \cup (V^{(i-1)} \setminus B^{(i-1)}) \cup (H^{(i)} \setminus B^{(i)})$. Since the extensions are pairwise disjoint, there is an extension $E(\mathcal{P}^{(i)})$ containing at most three guards. If there are no guards in any of the bends $B^{(i-1)}, B^{(i)}$ it follows from Theorem 1 that three guards must be placed inside $\mathcal{P}^{(i)}$ at irrational coordinates, so assume that there is a guard in one or both of the bends. If the adjacent corridors $V^{(i-1)}$ and $H^{(i)}$ are long enough and thin enough, a guard in the bends $B^{(i-1)}$ and $B^{(i)}$ cannot see any left corner of any of the vertical pockets of $\mathcal{P}^{(i)}$, any point in a triangular pocket, or any point in a horizontal pocket. Hence, all the features of $\mathcal{P}^{(i)}$ that enforce the irrationality of the guards are unseen by the guards in the bends and it follows that there must be irrational guards in $\mathcal{P}^{(i)}$. Therefore, at least $4n$ guards are needed if we require them to be rational. Similarly as in the proof of Theorem 1, we can show that $4n$ rational guards are enough to guard \mathcal{P}_n . \square

5 Rectilinear Polygon

Figure 8 depicts a rectilinear polygon \mathcal{P}_R with corners at rational coordinates that can be guarded by 9 guards, but requires 10 guards if we restrict the guards to points with rational coordinates. Before the formal proof, we want to give the reader a short overview. The construction of \mathcal{P}_R starts with the polygon \mathcal{P} from Theorem 1. We will extend the non-rectilinear parts by “equivalent” rectilinear parts, colored gray in the figure. The rectilinear pockets will be constructed in such a way, that each of them will require at least one guard in the interior. Additionally, if the interior of each pocket contains only one guard, then these guards must be placed at specific positions, making the area not seen by these six additional guards exactly the polygon \mathcal{P} described in Section 2 (the white area in Figure 8). Thus, the remaining 3 guards must be placed at three irrational points by Theorem 1.

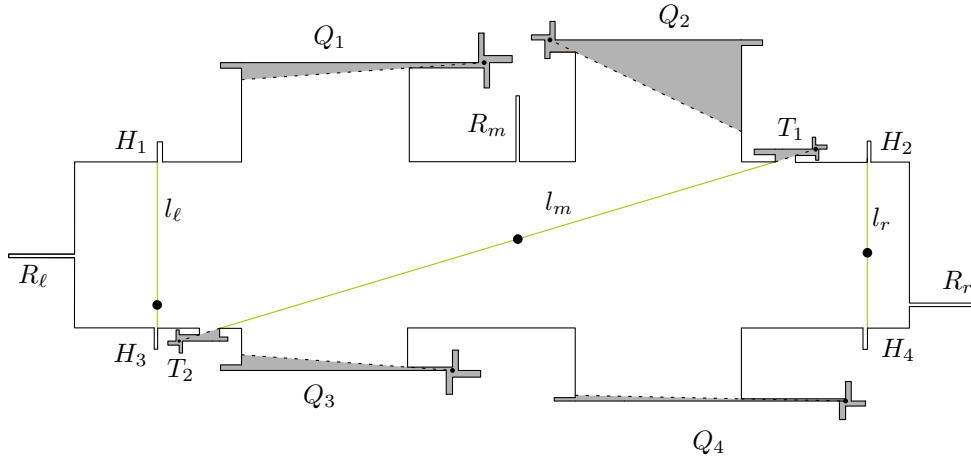


Figure 8: The rectilinear polygon \mathcal{P}_R can be guarded with 9 guards only when we allow placing guards at irrational points.

Proof of Theorem 3. We describe a polygon \mathcal{P}_R with vertices at integer coordinates that can be guarded by 9 guards with irrational coordinates, but needs 10 guards if only rational coordinates are allowed.

The construction of \mathcal{P}_R starts with the polygon \mathcal{P} from Theorem 1. We will replace the non-rectilinear parts by “equivalent” rectilinear parts, see Figure 8 for an illustration of the complete polygon \mathcal{P}_R and the notation therein. The additional areas need to be guarded by additional guards, as will be described later.

First, consider the triangular pockets of the polygon \mathcal{P} . These pockets have been added to enforce the guards to be on the lines l_ℓ, l_m, l_r . Four of these pockets, the ones corresponding to l_ℓ and l_r , can be easily replaced by corresponding rectilinear pockets denoted by H_1, H_2, H_3, H_4 , where three vertices of the new rectilinear pockets are the same as the vertices of the original triangular pockets. This does not work for the pockets corresponding to the line l_m , as this line is not axis-parallel, in particular, a guard on the line l_m would not see all of the interior of such rectangular pockets.

The two triangular pockets corresponding to l_m and the four quadrilateral pockets will be extended to new, more complicated pockets. Note that there are only two different kinds of pockets that need to be extended, *triangular pockets* and *quadrilateral pockets*, as pictured on the left of Figure 9 and Figure 10. Each triangular pocket is defined by three vertices and one of the sides of each pocket is not axis-parallel. Similarly, each quadrilateral pocket is defined by four vertices and one of the sides of each pocket is not axis-parallel.

Consider a pocket P , which needs to be extended in order to become rectilinear. Our extensions are pictured in the middle of Figure 9 and 10. The green area in the middle of Figure 9 and 10 is a newly-created pocket Q . For now, let us assume that Q does not intersect other parts of the polygon. The pocket Q satisfies the following properties (see the right pictures in Figure 9 and 10).

- (a) There are four points $p_1(Q), p_2(Q), p_3(Q), p_4(Q)$ within Q , such that each of them can only be seen by a guard, which is inside Q .
- (b) There exists exactly one point $q(Q)$ that can see all four points $p_1(Q), p_2(Q), p_3(Q), p_4(Q)$.
- (c) The point $q(Q)$ sees exactly the interior of Q .
- (d) All vertices of Q are rational.

We now show that a pocket Q satisfying all these properties can be constructed. First, we extend the non-axis-parallel edge of the pocket P in the direction outside the polygon and place a point $q = q(Q)$, with rational coordinates, on it. We let p_1, p_2, p_3, p_4 be points with rational coordinates directly above, to the right, below, and to the left of q , respectively. Then, we construct four rectilinear sub-pockets each with a vertex at one of the points p_1, p_2, p_3, p_4 , so that all these can be seen by q . These pockets can also be constructed with rational coordinates because q has rational coordinates. Clearly, we can choose the point q close enough to P so that the resulting pocket Q does not intersect the rest of the polygon.

Let \mathcal{P}_R be the constructed rectilinear polygon as pictured in Figure 8, where all triangular and horizontal pockets have been extended by rectilinear pockets. We have to show that \mathcal{P}_R can be guarded by 9 guards, but that we need 10 guards if we require the guards to be at rational coordinates. The underlying idea is that after an optimal placement of one guard in each of the six pockets that have been extended in order to become rectilinear, the remaining area that must be seen by the remaining guards is exactly the same as in the original polygon \mathcal{P} .

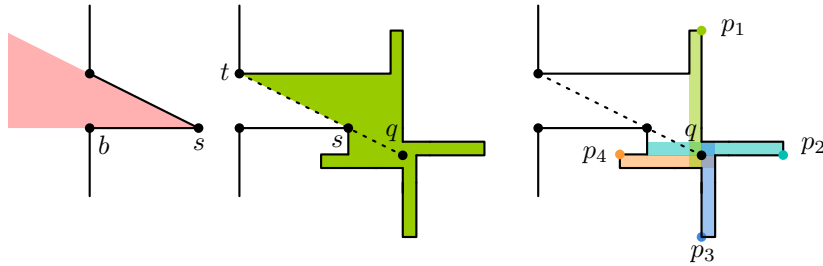


Figure 9: A triangular pocket is extended into a new rectilinear pocket.

We first present a solution with 9 guards when we are allowed to place guards at points with irrational coordinates. For this we place guards at the points $q(T_1), q(T_2), q(Q_1), q(Q_2), q(Q_3), q(Q_4)$ so that the interior of each of the pockets $T_1, T_2, Q_1, Q_2, Q_3, Q_4$ is seen, see Property (b) and (c). Then we cover the remaining part of the polygon with three irrational guards as described in the proof of Theorem 1.

It remains to show that 10 guards are required when we restrict the guards to have rational coordinates. Suppose for the purpose of contradiction that there is a solution with 9 rational guards. Note that

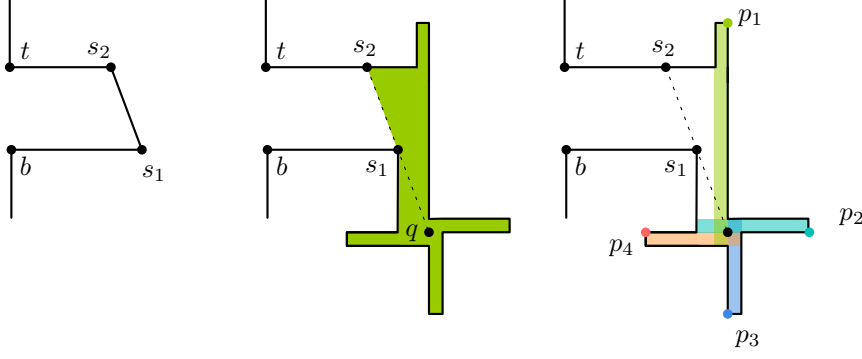


Figure 10: A quadrangular pocket is extended into a new rectilinear pocket.

there must be at least one guard in each pocket $Q \in \{T_1, T_2, Q_1, Q_2, Q_3, Q_4\}$ because of Property (a). We will now show that there must be at least three guards placed outside of $T_1 \cup T_2 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4$. First, notice that no guard placed in any of the pockets $T_1, T_2, Q_1, Q_2, Q_3, Q_4$ can see any of the following points: the top-left vertex of H_1 and H_2 , and the bottom-right vertex of H_3 and H_4 . To see these four points, at least two guards are needed. If there are only two guards, one of them must lie on l_ℓ , and the other on l_r . But then none of the guards placed on $l_\ell \cup l_r \cup T_1 \cup T_2 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ can see the top edge of the pocket R_m , and one more guard is needed. Therefore, at least three guards must be placed outside of $T_1 \cup T_2 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4$.

When only 9 guards are available, there must be exactly 3 guards outside the pockets $T_1, T_2, Q_1, Q_2, Q_3, Q_4$, and exactly one guard inside each pocket $Q \in \{T_1, T_2, Q_1, Q_2, Q_3, Q_4\}$. As each pocket Q contains exactly one guard, then this guard must be the point $q(Q)$ because of Property (a) and (b). Let \mathcal{P}_R^* be the area unseen by the guards within the six pockets. This polygon is exactly \mathcal{P} , and by Theorem 1 the unique solution with three guards is irrational. A polygon with integer coordinates can then be obtained by multiplying all coordinates with the least common multiple of all denominators of the coordinates. \square

6 Future Work

One of the most prominent open questions related to the art gallery problem is whether the problem is in NP. Recently, some researchers popularized an interesting complexity class, called $\exists\mathbb{R}$, being somewhere between NP and PSPACE [8, 9, 23, 27]. Many geometric problems for which membership in NP is uncertain have been shown to be complete for the complexity class $\exists\mathbb{R}$. Famous examples are: order type realizability, pseudoline stretchability, recognition of segment intersection graphs, recognition of unit disk intersection graphs, recognition of point visibility graphs, minimizing rectilinear crossing number, linkage realizability. This suggests that there might indeed be no polynomial sized witness for any of these problems as this would imply $\text{NP} = \exists\mathbb{R}$. It is an interesting open problem whether the art gallery problem is $\exists\mathbb{R}$ -complete or not.

The irrational coordinates of the guards in our examples are all of degree 2, i.e., they are roots in second-degree polynomials with integer coefficients. We would like to know if polygons exist where irrational numbers of higher degree are needed in the coordinates of an optimal solution.

We have constructed a simple polygon requiring three guards placed at points with irrational coordinates. It is a natural question whether there exists a polygon which can be guarded by two guards only if they are placed at points with irrational coordinates.

We show that there exists polygons for which $|OPT_{\mathbb{Q}}| \geq \frac{4}{3}|OPT|$. It follows from the work by Bonnet and Miltzow [5] that it always holds that $|OPT_{\mathbb{Q}}| \leq 9|OPT|$. It is interesting to see if any of these bounds can be improved.

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A Computations

```
def colinear(A,B,C):
    return Matrix([[A[0],A[1],1],[B[0],B[1],1],[C[0],C[1],1]]).determinant()

R.<t,p1,q1,p2,q2,x,y> = QQ[]

eq1 = ideal(
    colinear((2,t),(4,4),(p1,q1)),
    colinear((2,t),(4,0),(p2,q2)),
    colinear((4,280/47),(p1,q1),(8,294/47)),
    colinear((4,-12/19),(p2,q2),(8,-18/19)),
    colinear((p1,q1),(8,4),(x,y)),
    colinear((p2,q2),(8,0),(x,y))
    ).elimination_ideal([t,p1,q1,p2,q2]).gens()[0]

eq2 = ideal(
    colinear((19,t),(16,4),(p1,q1)),
    colinear((19,t),(16,0),(p2,q2)),
    colinear((16,1776/375),(p1,q1),(12,2486/375)),
    colinear((16,-36/21),(p2,q2),(12,-34/21)),
    colinear((p1,q1),(12,4),(x,y)),
    colinear((p2,q2),(12,0),(x,y))
    ).elimination_ideal([t,p1,q1,p2,q2]).gens()[0]

print eq1
print eq2
```